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CONTRACT THEORY

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1

Hidden Income

Introduction and overview

This Chapter introduces some models of hidden income.

Capital structure

Example 1.1 *Perfect markets*

We solve the problem of a risk averse entrepreneur with initial wealth w and utility function $u(c)$ where $u', -u'' > 0$. The entrepreneur is raising funds from an external financier who is risk neutral with respect to the outcome of the firm's project.[1] *The entrepreneur has access to an investment project returning $y = z \cdot r(a)$, where a is the total amount invested in the project, z is a random variable, $z = z_i$ with probability π_i, $r(a)$ is concave, $r, r', -r'' > 0$. The external financier's gross opportunity cost of funds is R.*[2]

The entrepreneur makes a take-it-or-leave-it offer to the external financier, who is competing with other financiers. Denoting repayments to the external financier by t_i, [3] *the entrepreneur solves the following programme*

$$\max_{c,t,a} \mathbb{E}[u(c)]$$

subject to the budget constraints

$$z_i r(a) \geq c_i + t_i \qquad \forall\ i,$$

and the financier's participation constraint

$$\mathbb{E}[t] \geq (a - w)R.$$

a. Verify that the objective function is concave[4] *and the constraints are convex.*

b. Solve the model using a Lagrangian.

c. What is the variance of the entrepreneur's consumption? Explain.

[1] This assumption of the risk neutrality or outside investors is motivated by a diversification argument: the individual entrepreneur's project is only a small part of the external financier's portfolio. Generalising these models to account for aggregate risks, to which the external financier is averse, opens up interesting questions that we will not consider here.

[2] This is the gross return they would earn on the next best alternative investment.

[3] The notation t comes from the link to applications in taxation.

[4] which implies that the negation of the objective function, which would be minimised in the standard form of Programme **??**, is convex.

d. *Verify and explain why* $\mathbb{E}[zr'(a)] = R$.

Solution 1.1 *Left to student as an exercise.*

Example 1.2 *Hidden income*

As in Example 1.1, but now assume that only the entrepreneur can observe the technology shock z*. By the revelation principle, there exists an optimal contract that elicits truth telling as an equilibrium action.*

We now have the incentive compatibility constraints,

$$u(c_i) \geq u(c_j + (z_i - z_j)r(a)) \qquad \forall\, i, j.$$

The left hand side is the utility attained by the entrepreneur recieving technology shock z_i *and reporting truthfully. The right hand side is the utility attained by the same entpreneur who falsely reports* j*: their consumption bundle is the consumption that would be due to a truthful reporting agent earning* j*, plus the "hidden income"* $(z_i - z_j)r(a)$.

Trick. If consumption is bounded below, we can just focus on the *downward* incentive constraints (ie. high income agents falsely reporting low incomes). To see this, apply a lottery where with probability ϵ, agents must pay their income minus some small adjustment δ. Low income agents cannot afford this high repayment and therefore they cannot credibly report high incomes.

a. *Verify that the incentive constraints are convex.*

b. *Show that repayments under the optimal contract are not contingent on the state* ($t_i = t_j\ \forall\, i, j$).

c. *Solve the entrepreneur's problem using a Lagrangian.*

d. *What is the variance of the entrepreneur's consumption? Explain.*

e. *Verify and explain why* $\mathbb{E}[zr'(a)] > R$.

Solution 1.2 *Left to student as an exercise.*

Auditing

We start with two irrelevance results, cases where we can elicit truthful reporting of income while auditing with arbitrarily low probability (and thereby at arbitrarily low monitoring cost). The first example is an example of *maximal deterrence*, the second could be called maximal rewards.

Example 1.3 *Consider a hidden information model with an agent who has a utility function* u *that is unbounded below:* $Ran(u) = (-\infty, \infty)$.[5] *There is an audit technology that reveals the true income of the agent at cost* κ.

Contracts specify repayment t_i *if the agent is not audited. If the agent is audited and has been found to have reported their income truthfully they receive rebate* r_i*. If they are found to have falsely reported income* y_j *then they pay fine* f_{ij}.

The principal solves for the optimal repayments t, r, f *and the optimal audit probabilities* q *to maximise net revenue subject to a lower bound on the agent's expected utility of* $\underline{u}$.

[5] The HARA class of utility functions, which inclues log and CRRA, have this property. As does risk neutrality if there is no non-negativity or limited liability constraint.

$$\max_{t,q,r,f} \sum_i \pi_i \left(t_i - q_i(\kappa + r_i)\right)$$

subject to the participation constraint,

$$\sum_i \pi_i(1-q_i)u(y_i - t_i) + q_i u(y_i - t_i + r_i) \geq \underline{u},$$

the incentive constraints,

$$(1-q_i)u(y_i - t_i) + q_i u(y_i - t_i + r_i) \geq (1-q_j)u(y_i - t_j) + q_j u(y_i - t_j - f_{ij}) \qquad \forall\, i > j.$$

Does an optimal contract exist?

Solution 1.3 *Consider the following contract:*

$$t_i = y_i - c(\underline{u}),$$

$$r_i = 0,$$

where $c(u) = u^{-1}(u)$. These conditions imply that $u(y_i - t_i), u(y_i - t_i + r_i) = \underline{u}$. The incentive constraint becomes

$$\underline{u} \geq (1-q_j)u(c(\underline{u}) + y_i - y_j) + q_j u(y_i - t_j - f_{ij}).$$

For any positive q_i, there exists f_{ij} that satisfies the incentive constraint, given that utility u is unbounded below. This contract has full consumption risk insurance, and arbitrarily low audit costs. The supremum is not obtainable. In the limit, the audit probability q_i approaches zero, but never reaches zero. Optimal contracts do not exist.

So, that was maximal deterrence: we economised on audit costs while still ensuring the incentive compatibility of truth-telling by setting audit probabilities arbitrarily low and penalties arbitrarily high. While we cannot always apply arbitrarily large penalties, the desire to do so is pretty difficult to avoid in contract theory models.

Example 1.4 *Maximum rewards. Border and Sobel [1987] present a model of tax extraction with costly audits.*[6] *Consider a tax authority who seeks to maximise revenue net of audit costs. The taxpayer reporting income y_i initially pays tax t_i. If they are audited, then they recieve rebate r_i if found to have truthfully reported their income, or else they pay fine f_{ji} if they are found to have actually received income y_j.*

The taxpayer is risk neutral but has reservation utility $\underline{u}$ and their consumption is bounded below by zero.[7] *The authors appeal to the 1954 Akira Kurosawa film Seven Samurai. The bandits demand tribute from the villagers, and must pay a cost to extract the tribute by force if it is not willingly offered (the audit cost). But, if the bandits demand too much, the villagers will hire Samurai to protect them from the bandits (the utility attained in the case*

[6] Kim C. Border and Joel Sobel. Samurai accountant: A theory of auditing and plunder. *Review of Economic Studies*, 54 (4):525–540, 1987

[7] This constraint is referred to as the *limited liability constraint*.

of hiring the Samurai and successfully defending the village from the bandits is the reservation utility).

$$\max_{t,q,r,f} \sum_i \pi_i \left(t_i - q_i(\kappa + r_i)\right)$$

subject to the participation constraint,

$$\sum_i \pi_i \left(y_i - t_i + q_i r_i\right) \geq \underline{u},$$

the incentive constraints,

$$y_i - t_i + q_i r_i \geq y_i - t_j - q_j f_{ij} \qquad \forall\, i > j,$$

the limited liability constraint,

$$y_i - t_j - f_{ij} \geq 0 \quad \forall\, i > j,$$

and the non-negativity constraints

$$0 \leq q_i \leq 1 \quad \forall\, i.$$

Consider a contract with $t_k < y_k$ for some $1 < k < n$, where there exists i with downward incentive constraint $y_i - t_i + q_i r_i \geq y_i - t_k - q_k f_{ik}$ binding, and where $t_k < y_k$.

a. *Show that if $q_k > 0$, the incentive constraint can be relaxed without the principal incurring any additional audit cost while retaining the same level of expected utility and retaining the incentives for agent earning k to report truthfully by increasing t_k to $t'_k := y_k$ and increasing r_k to $r'_k := r_k + (y_k - t_k)/q_k$.*

b. *Following the perturbation described in part (a), we have $q_k > 0, r'_k > 0, b'_k = y_k$. Show that the principal can reduce audit costs without violating the incentive or participation constraints by reducing q_k and increasing r_k.*

Solution 1.4 *Left to student as an exercise.*

Parts (a) and (b) of Example 1.4 together show us that we should reward agents who we have verified to be truth-telling (part a) and that if agents are risk neutral, we can reduce audit costs to arbitrarily low levels by applying stochastic audits with low probability. When we add risk aversion, these lotteries are costly: increasing consumption risk means increasing the resource cost of attaining a pre-determined level of expected utility.

Example 1.5 *Duncan and Nolan [2017] study a hidden information model with a risk averse entrepreneur and costly audits that generate noisy signals* *of the true income of the entrepreneur.*[8] *The authors verify the Mangasarian-Fromowitz constraint qualification for the consumption allocation subproblem, which means that we can use the standard Lagrangian / Kuhn-Tucker approaches to obtain first order necessary conditions.*

Assume that the external financier cannot observe income y*, but instead that they can observe some signal* σ *which is drawn from a distribution that is affected by the true income* y*. That is, there exists* s, t, l, k *such that*

$$\frac{P(\sigma_s|y_l)}{P(\sigma_t|y_l)} \neq \frac{P(\sigma_s|y_k)}{P(\sigma_t|y_k)}.$$

Assume $P(\sigma_s|y_l) > 0 \ \forall \ s, l$*.*

Contracts specify consumption $c_i(\sigma)$ *as a function of the income state* i *and the signal* σ, $c_i(\sigma)$*. Again, assume that the entpreneur is merely solving for consumption allocations, contingent on incentivising truthful reports and satisfying the participation constraint of the external financier. Let* $\underline{z}$ *capture the lower bound on expected repayments implied by the external financier's opportunity cost of funds. The objective function of the entrepreneur is*

$$\max_{z_i(\sigma)} \sum_{i,s} \pi_i P(\sigma_s|y_i) u(c_i(\sigma_s))$$

subject to the participation constraint

$$\sum_{i,s} \pi_i P(\sigma_s|y_i)(y_i - c_i(\sigma)) \geq \underline{z},$$

and the incentive (truth-telling) constraints

$$\sum_s P(\sigma_s|y_i) u(c_i(\sigma_s)) \geq \sum_s P(\sigma_s|y_i) u(c_j(\sigma_s) + y_i - y_j).$$

Assign λ *as the Kuhn-Tucker multiplier on the participation constraint, and* $\pi_i \mu_{ij}$ *as Kuhn-Tucker multiplier on the truth-telling constraint for agent recieving* i *but considering a report* j*. Show that the first order condition of this programme is*

$$u'(c_i(\sigma_s)) = \frac{1}{1 + \sum_{k<i} \mu_{ik}} \left[\lambda + \sum_{k>i} \mu_{ki} \frac{P(y_k|\sigma_s)}{P(y_i|\sigma_s)} u'(c_i(\sigma_s) + y_k - y_i) \right].$$

Comment on this result.

Solution 1.5 *Left to student as an exercise.*

[8] Alfred Duncan and Charles Nolan. Disputes, debt and equity. Studies in economics, School of Economics, University of Kent, 2017

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Hidden Effort - The Principal Agent Model

Introduction and overview

This Chapter introduces some models of hidden effort.

The principal-agent model.

A manager (the principal) is writing an incentive contract for an employee (the agent). The revenue generated for the firm is contingent on the employee's, action a, which affects the probability distribution of revenue outcomes $\theta \in \{\theta_1, \theta_2, ...\theta_n\}$.

$$\Delta_i(a) = P(\theta = \theta_i \mid a).$$

The principal is risk neutral with respect to the revenue generated by the individual agent; their net income in state i is given by

$$\theta_i - c_i(a).$$

The agent is risk averse with

$$v(c, e) = u(c) - e(a), \quad u', -u'' > 0, \; e > 0.$$

The agent likes consumption but dislikes effortful actions. The agent has an opportunity cost, an outside option that yields value $\bar{v} \in$ Ran (u).

Example 2.1 *Assume that the principal can observe a. This means that incentive contracts can specify consumption as a function of both the state i and the action a,* $c_i(a)$*. The principal's problem becomes*

$$\max_{c,a} \sum_i \Delta_i(a)(\theta_i - c_i(a)),$$

subject to the agent's participation constraint,

$$\sum_i \Delta_i(a) u(c_i(a)) - e(a) \geq \bar{v},$$

and the incentive compatibility constraint

$$\sum_i \Delta_i(a)u(c_i(a)) - e(a) \geq \sum_i \Delta_i(a')u(c_i(a')) - e(a') \quad \forall a' \neq a.$$

a. *Let a^* be the optimal level of effort. Show that for any allocation of consumption contingent on a^*, $c(a^*)$ that satisfies the participation constraint, there exist allocations $c(a')$ that leave the incentive compatibility constraint non-binding for all $a' \neq a^*$.*

b. *Show by contradiction that the optimal consumption allocations of the agent are not contingent on the state. That is, show that*

$$c_i(a) = c_j(a), \ \forall \ a, i, j.$$

Explain why this consumption allocation is consistent with first best allocative efficiency.

c. *Use the results from (a) and (b) to re-write the principal's problem. Write down an optimality condition for the action a. Relate the terms in the equation to concepts in allocative efficiency.*

Solution 2.1 *Left to student as an exercise.*

Example 2.2 *Assume that the principal cannot observe a. This means that incentive contracts can only specify consumption as a function of the state i, c_i. Also, assume that the principal is merely solving for consumption allocations, contingent on achieving a predetermined target action a^*.*

$$\max_c \sum_i \Delta_i(a^*)(\theta_i - c_i),$$

subject to the agent's participation constraint,

$$\sum_i \Delta_i(a^*)u(c_i) - e(a^*) \geq \bar{v},$$

and the incentive compatibility constraint

$$\sum_i \Delta_i(a^*)u(c_i) - e(a^*) \geq \sum_i \Delta_i(a')u(c_i) - e(a') \quad \forall a' \neq a^*.$$

Show that non-contingent incentive schemes are not incentive compatible. That is, there is not incentive compatible scheme with $c = c_i = c_j \ \forall \ i, j$.

Solution 2.2 *Left to student as an exercise.*

Example 2.3 *When incentive constraints contain concave utility functions entering with both positive and negative signs, it is not straightforward to prove that these constraints are convex (sometimes they are non-convex). Grossman and Hart*[1] *propose a clever substitution outlined below, that can often be used to verify convexity in some cases.*

[1] Sanford Grossman and Oliver Hart. An analysis of the principal-agent problem. *Econometrica*, 51(1):7–45, 1983

Assume that the principal cannot observe a. This means that incentive contracts can only specify consumption as a function of the state i, c_i. Also, assume that the principal is merely solving for consumption allocations, contingent on achieving a predetermined target action a^*.

$$\max_c \sum_i \Delta_i(a^*)(\theta_i - c_i),$$

subject to the agent's participation constraint,

$$\sum_i \Delta_i(a^*)u(c_i) - e(a^*) \geq \bar{v},$$

and the incentive compatibility constraint

$$\sum_i \Delta_i(a^*)u(c_i) - e(a^*) \geq \sum_i \Delta_i(a')u(c_i) - e(a') \quad \forall a' \neq a^*.$$

Apply the substitutions $u_i := u(c_i)$, and $c(u_i) := c_i$. Then, rewrite the problem so that the principal chooses utility allocations u_i directly, rather than consumption allocations c_i. Show that in the revised programme, the objective function (expressed as a maximand) is concave and the constraints are affine in the choice variables.

Solution 2.3 *Left to student as an exercise.*

Example 2.4 *Consider a model where there are two revenue states. Show using a diagram, that for a given expected utility $\mathbb{E}[u] = \bar{u}$, the expected marginal cost of utility is greater when consumption is risky than when consumption is constant.*

Given your answers to the earlier examples, what do you think this will mean for the optimal actions in the case where actions are not observed by the principal?

Solution 2.4 *Left to student as an exercise.*

Example 2.5 *Sufficient statistics.*

Using the revised formulation of the principal's consumption allocation sub-problem from Example 2.3, we can exploit the convexity of the problem and use the standard Lagrangian / Kuhn-Tucker approaches.

Assume that the principal cannot observe a, but instead that they can observe some signal σ which is drawn from a distribution that is affected by the action a. That is, there exists s, t, l, k such that

$$\frac{P(\sigma_s|a_l)}{P(\sigma_t|a_l)} \neq \frac{P(\sigma_s|a_k)}{P(\sigma_t|a_k)}.$$

Assume $P(\sigma_s|a_l) > 0 \; \forall \; s, l$.

Incentive contracts specify consumption as a function of the state i and the signal σ, $c_i(\sigma)$. Again, assume that the principal is merely solving for *consumption allocations, contingent on achieving a predetermined target action a^*.*

$$\max_c \sum_i \Delta_i(a^*) \left(\theta_i - \sum_s P(\sigma_s|a^*) c_i(\sigma_s) \right),$$

subject to the agent's participation constraint,

$$\sum_{i,s} \Delta_i(a^*) P(\sigma_s|a^*) u(c_i(\sigma_s)) - e(a^*) \geq \bar{v},$$

and the incentive compatibility constraint

$$\sum_{i,s} \Delta_i(a^*) P(\sigma_s|a^*) u(c_i(\sigma_s)) - e(a^*) \geq \sum_{i,s} \Delta_i(a_l) P(\sigma_s|a_l) u(c_i(\sigma_s)) - e(a_l) \quad \forall a_l \neq a^*.$$

Apply the substitutions $u_i(\sigma_s) := u(c_i(\sigma_s))$, and $c(u_i(\sigma_s)) := c_i(\sigma_s)$. Then, rewrite the problem so that the principal chooses utility allocations $u_i(\sigma_s)$ directly, rather than consumption allocations $c_i(\sigma_s)$.

Assigning the Kuhn-Tucker multipliers λ, μ_l to the participation constraint and the incentive constraints respectively, derive and comment on the following first order necessary condition:

$$c'(u_i(\sigma_s)) = \lambda + \sum_l \mu_l \left[1 - \frac{\Delta_i(a_l) P(\sigma_s|a_l)}{\Delta_i(a^*) P(\sigma_s|a^*)} \right]$$

Solution 2.5 *Left to student as an exercise.*

3
Bibliography

Kim C. Border and Joel Sobel. Samurai accountant: A theory of auditing and plunder. *Review of Economic Studies*, 54(4):525–540, 1987.

Alfred Duncan and Charles Nolan. Disputes, debt and equity. Studies in economics, School of Economics, University of Kent, 2017.

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